A NEW RESTRICTION FOR INITIALLY STRESSED ELASTIC SOLIDS

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Summary

We introduce a fundamental restriction on the strain energy function and stress tensor for initially stressed elastic solids. The restriction applies to strain energy functions W that are explicit functions of the elastic deformation gradient $\mathbf{F}$ and initial stress $\boldsymbol{\tau}$, that is $W := W(\mathbf{F}, \boldsymbol{\tau})$. The restriction is a consequence of energy conservation and ensures that the predicted stress and strain energy do not depend upon an arbitrary choice of reference configuration. We call this restriction *initial stress reference independence* (ISRI). It transpires that *most* strain energy functions found in the literature do not satisfy ISRI, and may therefore lead to unphysical behaviour, which we illustrate through a simple example. To remedy this shortcoming, we derive three strain energy functions that *do* satisfy the restriction. We also show that using *initial strain* (often from a virtual configuration) to model initial stress leads to strain energy functions that automatically satisfy ISRI. Finally, we reach the following important result: ISRI reduces the number of unknowns in the linear stress tensor for initially stressed solids. This new way of reducing the linear stress may open new pathways for the non-destructive determination of initial stresses through ultrasonic experiments, among others.

1. Introduction

Materials in many contexts operate under a significant level of internal stress, which is often called *residual stress* if the material is not subjected to any external loading. Residual stress is desirable in many circumstances; for example, living matter uses residual stress to preserve ideal mechanical conditions for its physiological function (**1**, **2**). In manufacturing, if residual stress is controlled, it can be used to strengthen materials such as turbine blades (**3**) and toughened glass (**4**); however, residual stress is often problematic as it can cause materials to fail prematurely (**5**, **6**). *Pre-stress* is another common term, which is often used to refer to internal stress caused by an external load (**7**, **8**, **9**, **10**). In this article, the term initial stress is used to describe any internal stress, irrespective of boundary conditions, and therefore encompasses *both* residual stress *and* pre-stress.

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In both industrial and biological contexts, the origin and extent of initial stresses are often unknown. One way to determine these stresses is by measuring how they affect the elastic response of the material. In metallurgy, it is well known that residual stress can be estimated by drilling small holes into a metal and observing how they change shape (**11**). Elastic waves are also used in many applications, since their behaviour is very sensitive to the initial stress in a material (**12**).

One alternative to link the response of a material to a very general dependence on the internal stress, therefore including initially stressed materials, is the implicit form of elasticity described by Rajagopal *et al*. (**13**, **14**, **15**), but this generality comes with the drawback of adding greater constitutive complexity. Explicit hyperelastic models are simpler and are accurate for many applications — the work of Hoger (**16**, **17**) and Man (**18**; **19**) has led to improved inverse methods for measuring initial stress (**20**, **21**, **22**, **23**, **24**, **25**) and monitoring techniques (**26**).

The mechanical properties of a hyperelastic material can be conveniently determined from its strain energy function W, which gives the strain energy per unit volume of the initially stressed *reference* configuration. In classical elasticity, W is a function of only the elastic deformation gradient $\mathbf{F}$ (that is $W := W(\mathbf{F})$). The simplest way to account for initial stresses is to allow W to depend on either the initial Cauchy stress tensor $\boldsymbol{\tau}$, or on an initial deformation gradient $\mathbf{F}_0$ from some stress-free configuration $\mathcal{B}_0$. For the first method, $W := W(\mathbf{F}, \boldsymbol{\tau})$ (**27**, **28**, **29**), whereas for the second, $W := J_0^{-1} W_0(\mathbf{F}\mathbf{F}_0)$ (**17**, **30**), where $J_0 = \det \mathbf{F}_0$ and W_0 is the strain energy per unit volume in $\mathcal{B}_0$. In both cases, $\mathbf{F}$ is the elastic deformation gradient from the initially stressed to the current configuration.

The two approaches each have relative advantages and disadvantages. If measuring the initial stress is the main goal, then using $W := W(\mathbf{F}, \boldsymbol{\tau})$ is the more direct method, but requires an extra restriction (which is presented below). It is also the more useful form when the initial stress is known or postulated *a priori* — by assuming that the stress gradient in an arterial wall is homogeneous (**31**), for example. If $W := J_0^{-1} W_0(\mathbf{F}\mathbf{F}_0)$, then the classical theory of non-linear elasticity can be used (by taking $\mathcal{B}_0$ as the reference configuration), and ISRI is automatically satisfied. This form is more useful when a stress-free configuration is known, or when the exact form of the initial stress is not important. The two approaches are not equivalent because it is not always possible to deduce $\mathbf{F}_0$ from $\boldsymbol{\tau}$ explicitly, as they are related by the equilibrium equation of the initially stressed configuration, which is a non-linear partial differential equation in $\mathbf{F}_0$. We discuss initially strained models in Section 3.

The primary purpose of this article is to deduce a fundamental restriction on $W := W(\mathbf{F}, \boldsymbol{\tau})$, and discuss its consequences. To motivate the need for a new restriction, we show how a simple uniaxial deformation can lead to unphysical results when this restriction is ignored in Section 2.1. In Section 2.2, we derive this restriction, which follows from the fact that elastic deformations conserve energy, and we call it *initial stress reference independence* (ISRI), for reasons that will be clarified later. We assume the only source of anisotropy is the initial stress, though a more general form of ISRI could also be deduced for materials that include other sources of anistropy.

It transpires that it is not easy to choose a strain energy function that satisfies ISRI. In fact, almost every strain energy function found by the authors in the literature to date does not satisfy it, in both finite elasticity **29**, **32**, **33**, **34**, **35**, **36**) and linear elasticity (**19**, **28**). To the authors' knowledge, the only existing strain energy function that does satisfy ISRI is that derived in (**31**), which is an initially stressed incompressible neo-Hookean solid, as discussed in Section 2.3. To address this lack of valid models, we present two new strain energy functions that satisfy ISRI in Section 2.4. In Section 3, we discuss strain energy functions based on *initial strain*, and show that they automatically satisfy ISRI in Section 3.1.

The equations associated with *small* deformations of initially stressed solids are much simpler than their finite strain analogues. This makes them ideal for establishing methods to measure initial stress. An important consequence of ISRI is that it restricts the linearised elastic stress tensor $\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$, as we discuss in Section 4. For materials subjected to *small* initial stress, we use ISRI to reduce the number of unknowns in $\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$ in Section 4.3. The result is a reduced version of the stress tensor deduced in (**19**), which could ultimately improve the measurement of initial stress via ultrasonic experiments, among others.

In the literature, it is common to deduce the linear stress tensor $\delta\boldsymbol{\sigma}$ by considering an initial strain from a stress-free configuration (**37**, **38**, **39**). This approach is broadly called acousto-elasticity, and as discussed in Section 3, the resulting $\delta\boldsymbol{\sigma}$ automatically satisfies ISRI, but leads to an indirect connection between $\delta\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. In fact, acousto-elasticity was used by Tanuma and Man (**40**) to restrict the form of $\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$ when both strain and initial stress are small, which led them to our equation (4.36) (their equation (81)). In our approach we clarify that this equation must hold for every initially stressed elastic material, regardless of the origins of this stress.

2. ISRI

The mechanical properties of an elastic material can be determined from its strain energy function W, which gives the strain energy per unit volume of the reference configuration. For an initially stressed material, W can be expressed in terms of the deformation gradient $\mathbf{F}$ from the reference to the current configuration and $\boldsymbol{\tau}$, the Cauchy stress in the reference configuration, so that $W := W(\mathbf{F}, \boldsymbol{\tau})$. In general, W may also depend on position, but we omit this dependency for clarity. We call $\boldsymbol{\tau}$ the initial stress tensor and, when discussing consititutive choices, we will not require any specific boundary conditions in the reference configuration, in agreement with (**32**) (that is the boundaries can either be loaded or unloaded).

In what follows, we assume that $\mathbf{F}$ is within the *elastic* regime of the material, but make no assumptions about how the initial stress formed. The Cauchy stress tensor $\boldsymbol{\sigma}$ (**41**, **27**) for an initially stressed material is given by

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{\tau}) - p\mathbf{I}, \tag{2.1}$$

where $J = \det \mathbf{F}$, $\mathbf{I}$ is the identity tensor and p is zero if the material is compressible or, otherwise, is a Lagrange multiplier associated with the incompressibility constraint $\det \mathbf{F} = 1$. We define differentiation with respect to a second-order tensor as follows:

$$\left(\frac{\partial}{\partial \mathbf{P}}\right)_{ij} = \frac{\partial}{\partial \mathrm{P}_{ji}}. \tag{2.2}$$

Before moving on, we present an example where a specific choice of $W(\mathbf{F}, \boldsymbol{\tau})$ leads to two different stress responses for the same uniaxial deformation.

2.1 *Motivating example*

To study the influence of initial stress on the elastic response of a material, a simple strain energy function was postulated by Merodio *et al.* (**32**) as follows

$$W_{\mathrm{MOR}} = \frac{\mu}{2}\left(\mathrm{tr}(\mathbf{F}^{\mathrm{T}}\mathbf{F}) - 3\right) + \frac{1}{2}\left(\mathrm{tr}(\mathbf{F}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{F}) - \mathrm{tr}\,\boldsymbol{\tau}\right), \tag{2.3}$$

where μ is a material constant — a quantity that is inherently associated with the material and does not depend upon the reference configuration or level of residual stress, the superscript T indicates the transpose operator and tr the trace. As W_{MOR} is used for incompressible materials, the Cauchy stress (2.1) becomes

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{F}\mathbf{F}^{\mathrm{T}} + \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}. \tag{2.4}$$

Consider an initially stressed material described by Euclidean coordinates (X, Y, Z). Suppose the initial stress takes the form of a homogeneous tension T along the X axis, and that the material is subsequently stretched along the same axis, then the components of the deformation gradient and initial stress tensor are given by

$$\mathrm{F} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

where λ is the amount of stretch. Applying stress-free boundary conditions on the faces not under tension gives $p = \lambda^{-1}\mu$, which in turn leads to

$$\sigma_{11} := \sigma_{11}(\lambda, T) = \lambda^2(\mu + T) - \lambda^{-1}\mu, \tag{2.6}$$

which is the stress necessary to support any stretch λ given an initial tension T. We will now choose two different ways of achieving the same uniaxial stretch $\lambda = \widetilde{\lambda}$ that should, *but do not*, result in the same stress when using the strain energy function (2.3). First, we consider a direct application of the stretch $\lambda = \widetilde{\lambda}$ and assume that the initial tension is $T = \tau_0$. In this case,

$$\widetilde{\sigma}_{11} = \sigma_{11}(\widetilde{\lambda}, \tau_0) = \widetilde{\lambda}^2(\mu + \tau_0) - \widetilde{\lambda}^{-1}\mu. \tag{2.7}$$

We can also achieve the same stretch in two steps by taking $\widetilde{\lambda} = \widehat{\lambda}\overline{\lambda}$. That is, first we stretch by $\overline{\lambda}$ and then apply a further stretch $\widehat{\lambda}$, as shown in Fig. 1. Taking $\lambda = \overline{\lambda}$, and again using $T = \tau_0$, results in the stress

$$\overline{\sigma}_{11} = \sigma_{11}(\overline{\lambda}, \tau_0) = \overline{\lambda}^2(\mu + \tau_0) - \overline{\lambda}^{-1}\mu, \tag{2.8}$$

in the intermediate configuration. To further stretch the material, we take this intermediate configuration as our intially stressed reference configuration, where the initial tension is now $T = \overline{\sigma}_{11}$. Upon applying the second stretch $\widehat{\lambda}$, we obtain

$$\widetilde{\sigma}_{11} = \sigma_{11}(\widehat{\lambda}, \overline{\sigma}_{11}) = \widehat{\lambda}^2(\mu + \overline{\sigma}_{11}) - \widehat{\lambda}^{-1}\mu \tag{2.9}$$

$$= \widehat{\lambda}^2\overline{\lambda}^2(\mu + \tau_0) + \widehat{\lambda}^2\mu - \widehat{\lambda}^2\overline{\lambda}^{-1}\mu - \widehat{\lambda}^{-1}\mu. \tag{2.10}$$

Both (2.7) and (2.10) result from the same uniaxial deformation, so should be identical, but, upon substituting $\widetilde{\lambda} = \widehat{\lambda}\overline{\lambda}$ into (2.7), we find they are not.

If, instead of (2.4), we had used an initially strained model, for example an incompressible neo-Hookean model $W := \mu\,\mathrm{tr}(\mathbf{F}\mathbf{F}_0)/2$, then this unphysical result would not occur. However, as explained in the introduction, when the initial strain or stress are unknown, both $\boldsymbol{\tau}$ and $\mathbf{F}_0$ are unknown, and an explicit form $W := W(\mathbf{F}, \boldsymbol{\tau})$ leads to more direct connections between the elastic response and initial stress $\boldsymbol{\tau}$.

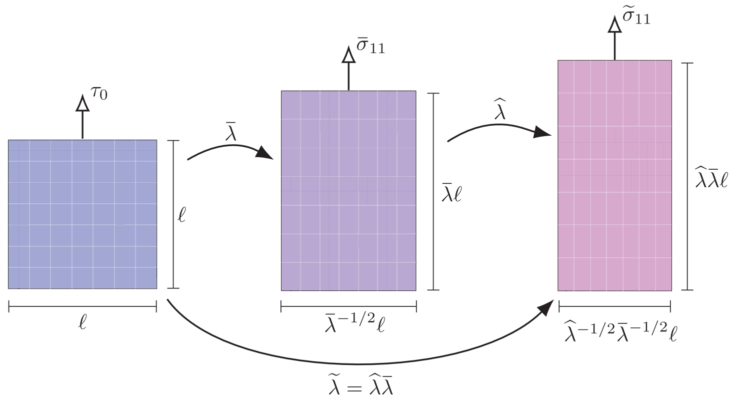

Fig. 1 Uniaxial deformation of an initially stressed cube (depth not illustrated), with sides of length ℓ, into a cuboid of height $\widehat{\lambda}\bar{\lambda}\ell$ and width (equal to depth) $\widehat{\lambda}^{-1/2}\bar{\lambda}^{-1/2}\ell$. The hollow arrows represent the stress applied to the top boundary. The uniaxial stretch $\widetilde{\lambda}$ is indicated by the bottom arrow. This stretch can also be achieved in two steps: first a stretch of $\bar{\lambda}$, then a further stretch of $\widehat{\lambda}$. The second of these stretches treats the middle configuration as its reference configuration. Both of these ways of achieving the same uniaxial stretch $\widehat{\lambda}\bar{\lambda}$ should require the same stress $\widetilde{\sigma}_{11}$ in the rightmost configuration

The unphysical behaviour illustrated by this example is typical of many of the strain energy functions of the form $W := W(\mathbf{F}, \boldsymbol{\tau})$ in the literature and highlights the need to restrict what forms are physically permissible. Therefore, in the following section, we present a restriction on $W(\mathbf{F}, \boldsymbol{\tau})$ that ensures that such unphysical behaviour does not occur.

2.2 *The restriction*

The elastic energy stored in a material should remain constant under a rigid motion, so $W(\mathbf{F}, \boldsymbol{\tau}) = W(\mathbf{QF}, \boldsymbol{\tau})$ for every proper orthogonal tensor $\mathbf{Q}$ (so that $\mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$ and $\det \mathbf{Q} = 1$). This identity can be used to show that W depends on $\mathbf{F}$ only through the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ (**41**), which we use to rewrite the Cauchy stress (2.1) as

$$\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = 2J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}(\mathbf{C}, \boldsymbol{\tau})\mathbf{F}^{\mathrm{T}} - p\mathbf{I}. \tag{2.11}$$

The presence of initial stress generally leads to an anisotropic material response, but for simplicity we assume that no other source of anisotropy is present. Referring to the three configurations shown in Fig. 2, let the strain energy per unit volume in $\widetilde{\mathcal{B}}$ be denoted by ψ. The strain energy stored as a result of the elastic deformation from $\mathcal{B}$ to $\widetilde{\mathcal{B}}$ should be the same as that due to successive elastic deformations from $\mathcal{B}$ to $\overline{\mathcal{B}}$, then from $\overline{\mathcal{B}}$ to $\widetilde{\mathcal{B}}$. In detail, taking $\mathcal{B}$ as the reference configuration, we conclude $\psi = \widetilde{J}^{-1}W(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau})$, where $\widetilde{J} = \widehat{J}\,\overline{J}$, $\widehat{J} = \det \widehat{\mathbf{F}}$ and $\overline{J} = \det \overline{\mathbf{F}}$, whereas if $\overline{\mathcal{B}}$ is taken as the reference configuration, we conclude $\psi = \widehat{J}^{-1}W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau}))$. Since these two quantities must be equal, we therefore have

$$\boxed{W(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau}) = \overline{J}W\left(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau})\right) \quad \text{for every} \quad \boldsymbol{\tau},\ \overline{\mathbf{F}} \text{ and } \widehat{\mathbf{F}},} \tag{2.12}$$

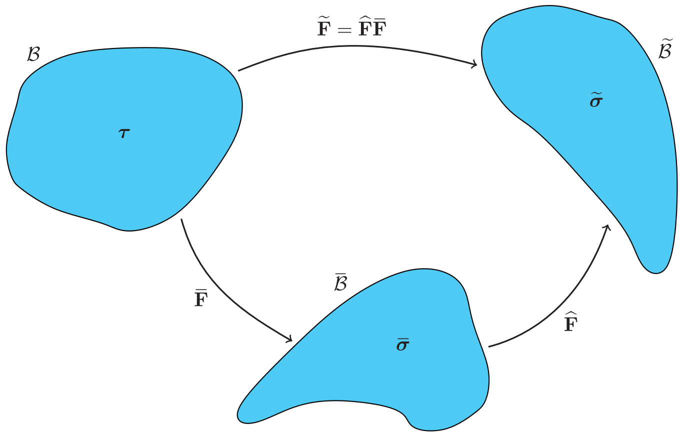

Fig. 2 Deformation of an initially stressed elastic solid. The stress and strain energy in $\widetilde{\mathcal{B}}$ should not depend on whether $\mathcal{B}$ or $\overline{\mathcal{B}}$ is taken as the reference configuration

where both $\overline{\mathbf{F}}$ and $\widehat{\mathbf{F}}$ are associated with *elastic* deformations (which may be constrained by incompressibility). We call this criterion *initial stress reference independence* (ISRI).

When $\overline{\mathbf{F}} = \mathbf{I}$, (2.12) reduces to $W(\widehat{\mathbf{F}}, \boldsymbol{\tau}) = W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\boldsymbol{I}, \boldsymbol{\tau}))$, which, from (2.11), is always satisfied if

$$\boldsymbol{\sigma}(\boldsymbol{I}, \boldsymbol{\tau}) = 2\frac{\partial W}{\partial \mathbf{C}}(\mathbf{I}, \boldsymbol{\tau}) - p\mathbf{I} = \boldsymbol{\tau}, \tag{2.13}$$

for every $\boldsymbol{\tau}$. We refer to this well-known restriction as *initial stress compatibility*. Additionally, if $\mathbf{F} = \mathbf{Q}$, where again $\mathbf{Q}$ is a proper orthogonal tensor representing a rigid body motion, then, using (2.11) and (2.13), we obtain $\boldsymbol{\sigma}(\mathbf{Q}, \boldsymbol{\tau}) = \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}}$. Using this result, along with $\overline{\mathbf{F}} = \mathbf{Q}$ in (2.12), we obtain

$$W(\widetilde{\mathbf{F}}, \boldsymbol{\tau}) = W\left(\widetilde{\mathbf{F}}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}}\right), \tag{2.14}$$

where $\widetilde{\mathbf{F}} = \widehat{\mathbf{F}}\overline{\mathbf{F}}$. The above identity is typically used for anisotropic materials (**42**) and can be used to derive the following 10 invariants (**29**)‡

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}[(I_1^2 - \operatorname{tr}(\mathbf{C}^2)], \quad I_3 = \det\mathbf{C}, \tag{2.15}$$

$$I_{\boldsymbol{\tau}_1} = \operatorname{tr}\boldsymbol{\tau}, \quad I_{\boldsymbol{\tau}_2} = \frac{1}{2}[(I_{\boldsymbol{\tau}_1}^2 - \operatorname{tr}(\boldsymbol{\tau}^2)], \quad I_{\boldsymbol{\tau}_3} = \det(\boldsymbol{\tau}), \tag{2.16}$$

$$J_1 = \operatorname{tr}(\boldsymbol{\tau}\mathbf{C}), \quad J_2 = \operatorname{tr}(\boldsymbol{\tau}\mathbf{C}^2), \quad J_3 = \operatorname{tr}(\boldsymbol{\tau}^2\mathbf{C}), \quad J_4 = \operatorname{tr}(\boldsymbol{\tau}^2\mathbf{C}^2), \tag{2.17}$$

‡ Note that the invariants I_{τ_1}, I_{τ_2} and I_{τ_3} are different from, but can be expressed as combinations of, those derived in (**29**).

though only nine of theses invariants are independent (**43**). Using these invariants, the Cauchy stress can be rewritten as

$$\begin{aligned}\boldsymbol{\sigma}(\mathbf{F},\boldsymbol{\tau}) = -p\mathbf{I} + \frac{1}{J}\Big(&2W_{I_1}\mathbf{B} + 2W_{I_2}(I_1\mathbf{B} - \mathbf{B}^2)\\ &+2I_3W_{I_3}\mathbf{I} + 2W_{J_1}\mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}} + 2W_{J_2}(\mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}\mathbf{B} + \mathbf{B}\mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}) + 2W_{J_3}\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^{\mathrm{T}}\\ &+2W_{J_4}(\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^{\mathrm{T}}\mathbf{B} + \mathbf{B}\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^{\mathrm{T}})\Big),\end{aligned} \tag{2.18}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green tensor, $W_{I_i} = \partial W/\partial I_i$ and $W_{J_j} = \partial W/\partial J_j$, with $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$. For an incompressible material $I_3 = 1$ and $W_{I_3} = 0$. Note that the Cauchy stress in a standard non-linear elastic material can be obtained from (2.18) simply by letting W depend only on the *strain* invariants I_1, I_2 and I_3.

By evaluating (2.18) at $\mathbf{F} = \mathbf{I}$ we obtain another form of the initial stress compatibility equation (2.13):

$$\boldsymbol{\tau} = \mathbf{I}(-\overset{\mathbf{I}}{p} + 2\overset{\mathbf{I}}{W}_{I_1} + 4\overset{\mathbf{I}}{W}_{I_2} + 2\overset{\mathbf{I}}{W}_{I_3}) + \boldsymbol{\tau}(2\overset{\mathbf{I}}{W}_{J_1} + 4\overset{\mathbf{I}}{W}_{J_2}) + \boldsymbol{\tau}^2(2\overset{\mathbf{I}}{W}_{J_3} + 4\overset{\mathbf{I}}{W}_{J_4}), \tag{2.19}$$

where the notation $\overset{\mathbf{I}}{\cdot}$ is used to denote that $\cdot$ is evaluated at $\mathbf{F} = \mathbf{I}$ *after* differentiation. Since this equation has to hold for *any* initial stress tensor $\boldsymbol{\tau}$, the initial stress compatibility condition is equivalent to

$$2\overset{\mathbf{I}}{W}_{I_1} + 4\overset{\mathbf{I}}{W}_{I_2} + 2\overset{\mathbf{I}}{W}_{I_3} = \overset{\mathbf{I}}{p}, \quad 2\overset{\mathbf{I}}{W}_{J_1} + 4\overset{\mathbf{I}}{W}_{J_2} = 1, \quad \overset{\mathbf{I}}{W}_{J_3} + 2\overset{\mathbf{I}}{W}_{J_4} = 0. \tag{2.20}$$

In the literature, W is often chosen as a simple function of the 10 invariants (2.15–2.17) that satisfy initial stress compatibility (2.20). However, it is highly unlikely that any W chosen *a priori* will satisfy ISRI (2.12).

A version of ISRI can also be stated in terms of the stress tensor, without reference to a strain energy function. To do so, let us assume the internal stress is given by some constitutive choice $\boldsymbol{\sigma} := \boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$, then using reasoning similar to that which led to (2.12) we find that

$$\boxed{\boldsymbol{\sigma}(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau}) = \boldsymbol{\sigma}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau})), \quad \text{for every } \boldsymbol{\tau},\ \overline{\mathbf{F}} \text{ and } \widehat{\mathbf{F}}.} \tag{2.21}$$

This restriction states that the Cauchy stress in $\widetilde{\mathcal{B}}$ should not change when a different reference configuration is selected. As the above is stated solely in terms of stress tensors, it could be possible to extend ISRI to materials without an explicit strain energy function.

By choosing $\widehat{\mathbf{F}}\overline{\mathbf{F}} = \mathbf{I}$ and using (2.13), we obtain $\boldsymbol{\tau} = \boldsymbol{\sigma}(\overline{\mathbf{F}}^{-1}, \overline{\boldsymbol{\sigma}})$, where $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau})$. This restriction was derived in (**31**) and termed *initial stress symmetry*. It allowed a straightforward way to model the adaptive remodelling of living tissues such as arterial walls towards an ideal target stress (**44**, **45**). For more details see (**31**) and (**46**). To the authors' knowledge, the only strain energy function that does satisfy initial stress symmetry and ISRI is that derived in (**31**).

As demonstrated in Section 2.1, strain energy functions that do not satisfy ISRI may exhibit unphysical behaviour. We prove this in the following section, then derive two new strain energy functions that satisfy ISRI in Section 2.4.

2.3 *An incompressible strain energy function that satisfies ISRI*

In a recent paper, Gower *et al.* (**31**) proposed the strain energy function

$$W_{\mathrm{GCD}} = \frac{1}{2}(p_0(I_{\boldsymbol{\tau}_1}, I_{\boldsymbol{\tau}_2}, I_{\boldsymbol{\tau}_3})I_1 + J_1 - 3\mu), \tag{2.22}$$

where p_0 is a function of $I_{\boldsymbol{\tau}_1}, I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$ given by

$$p_0 = \frac{1}{3}\left(T_3 + \frac{T_1}{T_3} - I_{\boldsymbol{\tau}_1}\right), \tag{2.23}$$

$$T_1 = I_{\boldsymbol{\tau}_1}^2 - 3I_{\boldsymbol{\tau}_2}, \quad T_2 = I_{\boldsymbol{\tau}_1}^3 - \frac{9}{2}I_{\boldsymbol{\tau}_1}I_{\boldsymbol{\tau}_2} + \frac{27}{2}(I_{\boldsymbol{\tau}_3} - \mu^3), \tag{2.24}$$

$$T_3 = \sqrt[3]{\sqrt{T_2^2 - T_1^3} - T_2}. \tag{2.25}$$

One way to derive W_{GCD} is to rewrite an initially strained neo-Hookean strain energy function as an initially stressed strain energy function (**31**). An alternative derivation is given in Appendix 5. Using W_{GCD} in (2.18), the left side of (2.21) becomes

$$\boldsymbol{\sigma}(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau}) = p_0\widehat{\mathbf{F}}\overline{\mathbf{B}}\widehat{\mathbf{F}}^{\mathrm{T}} - \widetilde{p}\,\mathbf{I} + \widehat{\mathbf{F}}\overline{\mathbf{F}}\boldsymbol{\tau}\overline{\mathbf{F}}^{\mathrm{T}}\widehat{\mathbf{F}}^{\mathrm{T}}, \tag{2.26}$$

and the right side becomes

$$\boldsymbol{\sigma}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau})) = (p_1 - \overline{p})\widehat{\mathbf{B}} + p_0\widehat{\mathbf{F}}\overline{\mathbf{B}}\widehat{\mathbf{F}}^{\mathrm{T}} - \widehat{p}\,\mathbf{I} + \widehat{\mathbf{F}}\overline{\mathbf{F}}\boldsymbol{\tau}\overline{\mathbf{F}}^{\mathrm{T}}\widehat{\mathbf{F}}^{\mathrm{T}}, \tag{2.27}$$

where p_1 is the Lagrange multiplier associated with $\overline{\mathbf{F}}$. In Appendix 5 we show that $\overline{p} = p_1$, and therefore (2.27) reduces to

$$\boldsymbol{\sigma}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \hat{\boldsymbol{\tau}})) = p_0\widehat{\mathbf{F}}\overline{\mathbf{B}}\widehat{\mathbf{F}}^{\mathrm{T}} - \widehat{p}\,\mathbf{I} + \widehat{\mathbf{F}}\overline{\mathbf{F}}\boldsymbol{\tau}\overline{\mathbf{F}}^{\mathrm{T}}\widehat{\mathbf{F}}^{\mathrm{T}}. \tag{2.28}$$

Equation (2.21) then gives

$$\boldsymbol{\sigma}(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau}) = \boldsymbol{\sigma}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau})) \quad \Leftrightarrow \quad \widehat{p} = \widetilde{p}. \tag{2.29}$$

Since equations (2.26) and (2.28) have exactly the same functional form and they must be subjected to the same boundary conditions because they both represent the Cauchy stress in $\widetilde{\mathcal{B}}$, their Lagrange multipliers must be equal (i.e. $\widehat{p} = \widetilde{p}$). Therefore, W_{GCD} *does* satisfy ISRI.

2.4 *Two compressible strain energy functions that satisfy ISRI*

By using the same method as that used in Appendix 5 to derive W_{GCD}, we have derived two new strain energy functions for compressible materials. Both are based on compressible extensions of the neo-Hookean model:

$$W_{\mathrm{CNH1}} = \frac{\mu}{2}(I_1 - 3 - 2\log\sqrt{I_3}) + \frac{\lambda}{2}(\log\sqrt{I_3})^2, \tag{2.30}$$

and

$$W_{\mathrm{CNH2}} = \frac{\mu}{2}(I_1 - 3 - 2\log\sqrt{I_3}) + \frac{\lambda}{2}(\sqrt{I_3} - 1)^2, \tag{2.31}$$

where μ and λ are the ground state first and second Lamé parameters, respectively. The initially stressed strain energy functions corresponding to these are

$$W_{\mathrm{GSC1}} = \frac{q_1}{2}I_1 + \frac{J_1}{2} - \frac{\mu}{2K_1}\left(3 + 2\log(K_1\sqrt{I_3})\right) + \frac{\lambda}{2K_1}\left(\log(K_1\sqrt{I_3})\right)^2 \tag{2.32}$$

and

$$W_{\mathrm{GSC2}} = \frac{q_2}{2}I_1 + \frac{J_2}{2} - \frac{\mu}{2K_2}\left(3 + 2\log(K_2\sqrt{I_3})\right) + \frac{\lambda}{2K_2}\left(K_2\sqrt{I_3} - 1\right)^2, \tag{2.33}$$

where q_1, q_2, K_1 and K_2 are functions of $I_{\boldsymbol{\tau}_1}$, $I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$ and can be thought of as *initial stress parameters* defined implicitly by the equations

$$\frac{\mu^3}{K_1} = q_1^3 + q_1^2 I_{\boldsymbol{\tau}_1} + q_1 I_{\boldsymbol{\tau}_2} + I_{\boldsymbol{\tau}_3}, \qquad q_1 = \frac{1}{K_1}(\mu - \lambda\log K_1), \tag{2.34}$$

$$\frac{\mu^3}{K_2} = q_2^3 + q_2^2 I_{\boldsymbol{\tau}_1} + q_2 I_{\boldsymbol{\tau}_2} + I_{\boldsymbol{\tau}_3}, \qquad q_2 = \frac{\mu}{K_2} + \lambda(1 - K_2), \tag{2.35}$$

where the solutions for K_1 and K_2 should both be real and such that $K_1 \to 1$ and $K_2 \to 1$ when $\boldsymbol{\tau} \to \mathbf{0}$. The Cauchy stress tensors corresponding to these strain energy functions are, respectively,

$$\boldsymbol{\sigma}_{\mathrm{GSC1}} = \frac{1}{J}\left(q_1\mathbf{B} + \frac{1}{K_1}(\lambda\log(JK_1) - \mu)\mathbf{I} + \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}\right), \tag{2.36}$$

and

$$\boldsymbol{\sigma}_{\mathrm{GSC2}} = \frac{1}{J}\left(q_2\mathbf{B} + \left(\lambda(I_3K_2 - J) - \frac{\mu}{K_2}\right)\mathbf{I} + \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}\right). \tag{2.37}$$

These constitutive equations provide a simple way to study the effects of initial stress on any deformation.

3. Initially strained materials

Another way to model initial stress is via initial strain. This is normally done by including an initial deformation gradient $\mathbf{F}_0$ from some configuration $\mathcal{B}_0$ in the strain energy function $W := J_0^{-1}W_0(\mathbf{F}\mathbf{F}_0)$, where $J_0 = \det\mathbf{F}_0$ and W_0 is the strain energy per unit volume in $\mathcal{B}_0$. This representation of W is a consequence of both a fundamental covariance argument (**47**, **48**), and utilising a virtual stress-free configuration (**30**). The Cauchy stress tensor is then given by (**47**, **48**)

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}(\mathbf{F}\mathbf{F}_0) = J^{-1}J_0^{-1}\mathbf{F}\frac{\partial W_0}{\partial \mathbf{F}}(\mathbf{F}\mathbf{F}_0) - p\mathbf{I}. \tag{3.1}$$

Usually, $W_0(\mathbf{F}\mathbf{F}_0)$ is chosen such that $\mathcal{B}_0$ is stress-free, that is, $\boldsymbol{\sigma}(\mathbf{I}) = \mathbf{0}$. Assuming that the initial strain is the only source of anisotropy, the strain energy can be shown to depend only on the isotropic invariants of $\mathbf{F}_0^{\mathrm{T}}\mathbf{C}\mathbf{F}_0$:

$$\widehat{I}_1 = \mathrm{tr}(\mathbf{F}_0^{\mathrm{T}}\mathbf{C}\mathbf{F}_0), \quad \widehat{I}_2 = \frac{1}{2}(\widehat{I}_1^2 - \mathrm{tr}((\mathbf{F}_0^{\mathrm{T}}\mathbf{C}\mathbf{F}_0)^2)), \quad \widehat{I}_3 = \det(\mathbf{F}_0^{\mathrm{T}}\mathbf{C}\mathbf{F}_0), \tag{3.2}$$

so that $W := J_0^{-1}W_0(\widehat{I}_1, \widehat{I}_2, \widehat{I}_3)$. These strain energy functions automatically satisfy ISRI, as shown below in Section 3.1. An example of such a strain energy function is this initially strained form of the Mooney–Rivlin strain energy function:

$$W_0 = C_1(\widehat{I}_1\widehat{I}_3^{-1/3} - 3) + C_2(\widehat{I}_2\widehat{I}_3^{-2/3} - 3) + C_3(\widehat{I}_3^{-1/2} - 1)^2, \tag{3.3}$$

where C_1, C_2 and C_3 are material constants that must be chosen such that the body is stress free when $\mathbf{F} = \mathbf{F}_0 = \mathbf{I}$.

Taking W as a function of $\mathbf{F}$ and $\boldsymbol{\tau}$, or of $\mathbf{F}$ and $\mathbf{F}_0$, gives two different perspectives on the same phenomenon, each being useful in different circumstances. The former is more useful when the initial *stress* is known, whereas the latter is more useful when the initial *strain* can somehow be inferred.

3.1 *All initially strained materials satisfy ISRI*

We have discussed, in previous sections, that it is not easy to choose a function of the form $W := W(\mathbf{F}, \boldsymbol{\tau})$ that satisfies ISRI (2.12). Let us consider the case of initially strained materials with

$$W = W(\mathbf{F}, \boldsymbol{\tau}) := J_0^{-1}W_0(\mathbf{F}\mathbf{F}_0), \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\sigma}_0(\mathbf{F}_0). \tag{3.4}$$

We will prove that if $W = W(\mathbf{F}, \boldsymbol{\tau})$ is defined as above, and the function $\boldsymbol{\sigma}_0$ is invertible, it satisfies ISRI for *any* choice of $W_0(\mathbf{F}\mathbf{F}_0)$. First we assume that for any W_0 and initial stress $\boldsymbol{\tau}$ there is a deformation gradient $\mathbf{F}_0$[§] such that

$$\boldsymbol{\tau} = \boldsymbol{\sigma}_0(\mathbf{F}_0) \quad \text{where} \quad \boldsymbol{\sigma}_0(\mathbf{F}_0) = J_0^{-1}\mathbf{F}_0\frac{\partial W_0(\mathbf{F}_0)}{\partial \mathbf{F}_0} - p\mathbf{I}. \tag{3.5}$$

Next, we define an initially *stressed* strain energy function

$$W(\mathbf{F}, \boldsymbol{\tau}) = W(\mathbf{F}, \boldsymbol{\sigma}_0(\mathbf{F}_0)) := J_0^{-1}W_0(\mathbf{F}\mathbf{F}_0) \quad \text{for every } \mathbf{F} \text{ and } \boldsymbol{\tau}. \tag{3.6}$$

By substituting $\mathbf{F} = \widehat{\mathbf{F}}\overline{\mathbf{F}}$ into (3.6) we obtain

$$W(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau}) = J_0^{-1}W_0((\widehat{\mathbf{F}}\overline{\mathbf{F}})\mathbf{F}_0) = \overline{J}\,\overline{J}^{-1}J_0^{-1}W_0(\widehat{\mathbf{F}}(\overline{\mathbf{F}}\mathbf{F}_0)) = \overline{J}W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}_0(\overline{\mathbf{F}}\mathbf{F}_0)). \tag{3.7}$$

Then, using (3.1), we obtain

$$\boldsymbol{\sigma}_{\mathbf{0}}(\overline{\mathbf{F}}\mathbf{F}_0) = \overline{J}^{-1}J_0^{-1}\overline{\mathbf{F}}\frac{\partial W_0}{\partial \overline{\mathbf{F}}}(\overline{\mathbf{F}}\mathbf{F}_0) - p\mathbf{I}, \tag{3.8}$$

[§] For there to be a unique $\mathbf{F}_0$, for every $\boldsymbol{\tau}$, the strain energy W_0 needs to be rank-one convex (**49**) and some restrictions need to be made about the reference configuration of $\mathbf{F}_0$ (**30**).

and, since $J_0^{-1}W_0(\overline{\mathbf{F}}\mathbf{F}_0) = W(\overline{\mathbf{F}}, \boldsymbol{\tau})$,

$$\boldsymbol{\sigma}_{\mathbf{0}}(\overline{\mathbf{F}}\mathbf{F}_0) = \overline{J}^{-1}\overline{\mathbf{F}}\frac{\partial W}{\partial \overline{\mathbf{F}}}(\overline{\mathbf{F}}, \boldsymbol{\tau}) - p\mathbf{I}, \tag{3.9}$$

which, using (2.1), gives

$$\boldsymbol{\sigma}_{\mathbf{0}}(\overline{\mathbf{F}}\mathbf{F}_0) = \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau}). \tag{3.10}$$

Substituting the above into (3.7) we obtain $W(\widehat{\mathbf{F}\overline{\mathbf{F}}}, \boldsymbol{\tau}) = \overline{J}W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau}))$, which is the ISRI restriction (2.12).

While such strain energy functions are guaranteed to satisfy ISRI, it is not often possible to state their dependence on the stress invariants $I_{\boldsymbol{\tau}1}, I_{\boldsymbol{\tau}2}$ and $I_{\boldsymbol{\tau}3}$ *explicitly* (a notable exception being the strain energy function discussed in Section 2.3). Instead, it may be necessary to define that dependence *implicitly*, as is the case for the two models presented in Section 2.4.

4. Linear elasticity with initial stress

Elastic waves in solids are highly sensitive to initial stress, and linear elastic models fit measurements from currently-employed experimental techniques well. Our aim here is, in the long run, to improve these measurements by using a linearised version of ISRI (2.12).

In Section 4.1 we deduce the linearised stress without considering ISRI. Then, in Section 4.2, we calculate a linearised form of ISRI and discuss how to use it to restrict the linearised stress. Hoger (**16**, **50**), Man *et al*. (**18**, **19**) derived the equations for small initial stress, up to first order in $\boldsymbol{\tau}$. In (**19**) the authors remark that many experiments indicate that for small deformations the elastic stress depends linearly on the initial stress, at least for metals. Motivated by these observations, we linearise the elastic stress in both the elastic strain and initial stress in Section 4.3 and reach a reduced form for the stress (4.37) which adds a restriction to all previous models, to the authors' knowledge. The restriction (4.36) has been used before in the literature (see equation (81) from (**40**)) but was deduced from the context of acousto-elasticity.

4.1 *Linear elastic stress*

For a small elastic deformation, we can write the associated deformation gradient as $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$, where $\mathbf{u}$ is a small displacement. By Taylor series expanding the Cauchy stress (2.1) about $\mathbf{F} = \mathbf{I}$, the linearised Cauchy stress becomes

$$\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = \boldsymbol{\tau} + \overset{\mathbf{I}}{\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}}} : \nabla\mathbf{u} + \mathcal{O}((\nabla\mathbf{u})^2), \tag{4.1}$$

where we have exploited the fact that $\boldsymbol{\sigma}(\mathbf{I}, \boldsymbol{\tau}) = \boldsymbol{\tau}$ and we remind the reader that $\overset{\mathbf{I}}{\cdot}$ denotes that $\cdot$ is evaluated at $\mathbf{F} = \mathbf{I}$ *after* differentiation. We define

$$\left(\frac{\partial \mathbf{P}}{\partial \mathbf{Q}}\right)_{ijkl} = \frac{\partial P_{ij}}{\partial Q_{lk}} \qquad \text{and} \qquad (\mathcal{C} : \mathbf{P})_{ij} = \mathcal{C}_{ij\alpha\beta}P_{\beta\alpha}, \tag{4.2}$$

for any second-order tensors $\mathbf{P}$ and $\mathbf{Q}$ and fourth-order tensor $\boldsymbol{\mathcal{C}}$, using Einstein summation convention for the repeated dummy indices α and β. Using (2.11) and (4.2) it can be shown that

$$\frac{\overset{\mathbf{I}}{\partial\boldsymbol{\sigma}}}{\partial\mathbf{F}} : \mathbf{P} = \frac{\partial}{\partial\mathbf{F}}\left(2J^{-1}\mathbf{F}\frac{\partial W}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}\right)\bigg|_{\mathbf{F}=\mathbf{I}} : \mathbf{P} = \mathbf{P}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{P}^{\mathrm{T}} - \boldsymbol{\tau}\,\mathrm{tr}\,\mathbf{P} + 4\frac{\overset{\mathbf{I}}{\partial^2 W}}{\partial\mathbf{C}^2} : \mathbf{P}, \tag{4.3}$$

for every second-order tensor $\mathbf{P}$, where we have exploited the fact that $2\partial W/\partial\mathbf{C}|_{\mathbf{F}=\mathbf{I}} = \boldsymbol{\tau}$ from (2.13). We now introduce the linear strain and rotation tensors:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}) \qquad \text{and} \qquad \boldsymbol{\omega} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}), \tag{4.4}$$

respectively, which satisfy $\nabla\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$. Substituting $\boldsymbol{\omega}$ for $\mathbf{P}$ in (4.3), we obtain

$$\frac{\overset{\mathbf{I}}{\partial\boldsymbol{\sigma}}}{\partial\mathbf{F}} : \boldsymbol{\omega} = \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega}, \tag{4.5}$$

since $\mathrm{tr}\,\boldsymbol{\omega} = 0$ and

$$\left(\frac{\overset{\mathbf{I}}{\partial^2 W}}{\partial^2\mathbf{C}} : \boldsymbol{\omega}\right)_{ij} = \frac{\overset{\mathbf{I}}{\partial^2 W}}{\partial \mathrm{C}_{ji}\partial \mathrm{C}_{\alpha\beta}}\omega_{\alpha\beta} = -\frac{\overset{\mathbf{I}}{\partial^2 W}}{\partial \mathrm{C}_{ji}\partial \mathrm{C}_{\beta\alpha}}\omega_{\beta\alpha} \Rightarrow \frac{\overset{\mathbf{I}}{\partial^2 W}}{\partial^2\mathbf{C}} : \boldsymbol{\omega} = \mathbf{0}, \tag{4.6}$$

where we have used the fact that $\boldsymbol{\omega}^{\mathrm{T}} = -\boldsymbol{\omega}$ and $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$. Using (4.4) and (4.5) we can now rewrite (4.1) as

$$\delta\boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega} + \frac{\overset{\mathbf{I}}{\partial\boldsymbol{\sigma}}}{\partial\mathbf{F}} : \boldsymbol{\varepsilon} + \mathcal{O}((\nabla\mathbf{u})^2). \tag{4.7}$$

At this point, we do not yet know the form of $\partial\boldsymbol{\sigma}/\partial\mathbf{F}|_{\mathbf{F}=\mathbf{I}} : \boldsymbol{\varepsilon}$ explicitly. It could be calculated directly from (2.18); however, an alternative approach is to write it as a general rank two symmetric tensor in terms of $\boldsymbol{\tau}$ that is expanded up to first order in $\boldsymbol{\varepsilon}$:

$$\begin{aligned}\frac{\overset{\mathbf{I}}{\partial\boldsymbol{\sigma}}}{\partial\mathbf{F}} : \boldsymbol{\varepsilon} = {} & \alpha_1\boldsymbol{\varepsilon} + (\alpha_2\mathbf{I} + \alpha_3\boldsymbol{\tau} + \alpha_4\boldsymbol{\tau}^2)\,\mathrm{tr}(\boldsymbol{\varepsilon}) + (\alpha_5\mathbf{I} + \alpha_6\boldsymbol{\tau} + \alpha_7\boldsymbol{\tau}^2)\,\mathrm{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}) \\ & + \alpha_8(\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}) + \alpha_9(\boldsymbol{\varepsilon}\boldsymbol{\tau}^2 + \boldsymbol{\tau}^2\boldsymbol{\varepsilon}) + \mathcal{O}((\nabla\mathbf{u})^2),\end{aligned} \tag{4.8}$$

where α_i, $(i = 1, ..., 9)$ are, in general, functions of I_{τ_1}, I_{τ_2} and I_{τ_3}. Note that neither $\mathrm{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}^2)$, $\boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau}$, $\boldsymbol{\tau}^2\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau}^2$, nor any power of $\boldsymbol{\tau}$ higher than two is present because they can be written as combinations of the terms already included (see Appendix 5). For more details on linearising elasticity see (**50**, **51**, **52**, **29**).

We now seek to restrict the parameters $\alpha_1, ..., \alpha_9$. We begin by rearranging (4.3) and contracting it twice on the left with an arbitrary second-order tensor $\mathbf{Q}$, to obtain

$$4\mathbf{Q} : \frac{\overset{\mathbf{I}}{\partial^2 W}}{\partial\mathbf{C}^2} : \mathbf{P} = (\mathbf{Q} : \boldsymbol{\tau})\,\mathrm{tr}\,\mathbf{P} - \mathbf{Q} : (\mathbf{P}\boldsymbol{\tau}) - \mathbf{Q} : (\boldsymbol{\tau}\mathbf{P}^{\mathrm{T}}) + \mathbf{Q} : \frac{\overset{\mathbf{I}}{\partial\boldsymbol{\sigma}}}{\partial\mathbf{F}} : \mathbf{P}. \tag{4.9}$$

Since (4.9) must hold for any $\mathbf{P}$ and $\mathbf{Q}$, we can swap them to obtain

$$4\mathbf{P} : \overset{\mathbf{I}}{\frac{\partial^2 W}{\partial \mathbf{C}^2}} : \mathbf{Q} = \mathbf{P} : \boldsymbol{\tau} \operatorname{tr} \mathbf{Q} - \mathbf{P} : (\mathbf{Q}\boldsymbol{\tau}) - \mathbf{P} : (\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}}) + \mathbf{P} : \overset{\mathbf{I}}{\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}}} : \mathbf{Q}. \tag{4.10}$$

Now, due to the fact that

$$\left(\overset{\mathbf{I}}{\frac{\partial^2 W}{\partial^2 \mathbf{C}}} \right)_{ijkl} = \left(\overset{\mathbf{I}}{\frac{\partial^2 W}{\partial^2 \mathbf{C}}} \right)_{klij} \tag{4.11}$$

we must have

$$\mathbf{P} : \overset{\mathbf{I}}{\frac{\partial^2 W}{\partial \mathbf{C}^2}} : \mathbf{Q} = \mathbf{Q} : \overset{\mathbf{I}}{\frac{\partial^2 W}{\partial \mathbf{C}^2}} : \mathbf{P}, \tag{4.12}$$

for every $\mathbf{P}$ and $\mathbf{Q}$. Upon substituting (4.9) and (4.10) into (4.12), and assuming that $\mathbf{P}$ and $\mathbf{Q}$ are small and symmetric, so that (4.8) holds with $\mathbf{P}$ and $\mathbf{Q}$ substituted for $\boldsymbol{\varepsilon}$, we find that (4.12) can hold if and only if

$$\alpha_4 = \alpha_7 = 0 \quad \text{and} \quad \alpha_5 = \alpha_3 + 1. \tag{4.13}$$

Substituing the above into (4.7), we obtain a reduced expression for the stress:

$$\boxed{\begin{aligned} \delta\boldsymbol{\sigma} &= \boldsymbol{\tau} + \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega} + \mathbf{I}\operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}) + \alpha_1\boldsymbol{\varepsilon} + \alpha_2\mathbf{I}\operatorname{tr}(\boldsymbol{\varepsilon}) + \alpha_3\left(\boldsymbol{\tau}\operatorname{tr}(\boldsymbol{\varepsilon}) + \mathbf{I}\operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau})\right) \\ &\quad + \alpha_6\boldsymbol{\tau}\operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}) + \alpha_8(\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}) + \alpha_9(\boldsymbol{\varepsilon}\boldsymbol{\tau}^2 + \boldsymbol{\tau}^2\boldsymbol{\varepsilon}). \end{aligned}} \tag{4.14}$$

In Section 4.2, we discuss the linearised version of ISRI and its relationship to the linear stress tensor given in (4.14). When the initial stress is small, we are able to derive a closed-form expression for the linear stress that satisfies ISRI, as is shown in Section 4.3.

4.1.1 *Initially stressed neo-Hookean models.* As an aside, we note that if the stress tensors for the initially stressed neo-Hookean models given in (2.36) and (2.37) are expanded for small deformations, the resulting linear stress tensors have the above form with

$$\alpha_1 = \frac{2}{K_1}(\mu - \lambda \log K_1),\ \alpha_2 = \frac{\lambda}{K_1},\ \alpha_3 = -\alpha_8 = -1,\ \alpha_6 = \alpha_9 = 0, \tag{4.15}$$

for the first model, and

$$\frac{\alpha_1}{2} = \frac{\mu}{K_2} + \lambda(1 - K_2),\ \alpha_2 = \lambda(2K_2 - 1),\ \alpha_3 = -\alpha_8 = -1,\ \alpha_6 = \alpha_9 = 0, \tag{4.16}$$

for the second.

4.2 *The linearised equations of ISRI*

We now wish to consider the restrictions that are imposed by ISRI in the case of small deformations. We begin by differentiating (2.12) with respect to $\overline{\mathbf{F}}$ to obtain

$$\frac{\partial W}{\partial \mathbf{F}}(\widehat{\mathbf{F}}\overline{\mathbf{F}}, \boldsymbol{\tau})\widehat{\mathbf{F}} = \frac{\partial \overline{J}}{\partial \overline{\mathbf{F}}} W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau})) + \overline{J}\frac{\partial W}{\partial \boldsymbol{\sigma}}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau}))\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}}(\overline{\mathbf{F}}, \boldsymbol{\tau}), \tag{4.17}$$

where $\partial/\partial\mathbf{F}$ denotes partial differentiation with respect to the first argument of the function and $\partial/\partial\boldsymbol{\sigma}$ denotes partial differentiation with respect to the second. Evaluating (4.17) at $\widehat{\mathbf{F}} = \overline{\mathbf{F}} = \mathbf{I}$ and contracting twice on the right with the linear strain tensor $\boldsymbol{\varepsilon}$ gives

$$\boxed{\boldsymbol{\tau} : \boldsymbol{\varepsilon} = \operatorname{tr}\boldsymbol{\varepsilon}\, \overset{\mathbf{I}}{W} + \frac{\overset{\mathbf{I}}{\partial W}}{\partial \boldsymbol{\tau}} : \frac{\overset{\mathbf{I}}{\partial \boldsymbol{\sigma}}}{\partial \mathbf{F}} : \boldsymbol{\varepsilon} \quad \text{for every } \ \boldsymbol{\tau} \ \text{ and } \ \boldsymbol{\varepsilon},} \tag{4.18}$$

which was simplified using (2.13). One of the terms on the right side can be expanded using the chain rule as follows

$$\frac{\overset{\mathbf{I}}{\partial W}}{\partial \boldsymbol{\tau}} = \beta_1 \mathbf{I} + \beta_2 \boldsymbol{\tau} + \beta_3 \boldsymbol{\tau}^2, \tag{4.19}$$

where

$$\beta_1 = \frac{\overset{\mathbf{I}}{\partial W}}{\partial \operatorname{tr}\boldsymbol{\tau}} = \frac{\overset{\mathbf{I}}{\partial W}}{\partial I_{\boldsymbol{\tau}_1}} + I_{\boldsymbol{\tau}_1}\frac{\overset{\mathbf{I}}{\partial W}}{\partial I_{\boldsymbol{\tau}_2}} + I_{\boldsymbol{\tau}_2}\frac{\overset{\mathbf{I}}{\partial W}}{\partial I_{\boldsymbol{\tau}_3}} + \frac{\overset{\mathbf{I}}{\partial W}}{\partial J_1} + \frac{\overset{\mathbf{I}}{\partial W}}{\partial J_2}, \tag{4.20}$$

$$\beta_2 = 2\frac{\overset{\mathbf{I}}{\partial W}}{\partial \operatorname{tr}(\boldsymbol{\tau})} = -\frac{\overset{\mathbf{I}}{\partial W}}{\partial I_{\boldsymbol{\tau}_2}} - I_{\boldsymbol{\tau}_1}\frac{\overset{\mathbf{I}}{\partial W}}{\partial I_{\boldsymbol{\tau}_3}} + 2\frac{\overset{\mathbf{I}}{\partial W}}{\partial J_3} + 2\frac{\overset{\mathbf{I}}{\partial W}}{\partial J_4}, \tag{4.21}$$

$$\beta_3 = 3\frac{\overset{\mathbf{I}}{\partial W}}{\partial \operatorname{tr}(\boldsymbol{\tau}^3)} = \frac{\overset{\mathbf{I}}{\partial W}}{\partial I_{\boldsymbol{\tau}_3}}. \tag{4.22}$$

Using (4.14) and (4.19) and the Cayley–Hamilton theorem (see Appendix 5) we can rewrite the restriction (4.18) in the form

$$\operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}) = (\gamma_0 + \overset{\mathbf{I}}{W})\operatorname{tr}\boldsymbol{\varepsilon} + \gamma_1 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}) + \gamma_2 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}^2) \quad \text{for every } \ \boldsymbol{\tau} \ \text{ and } \ \boldsymbol{\varepsilon}, \tag{4.23}$$

where γ_0, γ_1 and γ_2 are functions of $\alpha_1, ..., \alpha_9, \beta_1, \beta_2, \beta_3, I_{\boldsymbol{\tau}_1}, I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$. Since (4.23) has to hold for every $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$ (for more details see the supplementary material of (**31**)), we obtain the three equations

$$\gamma_0 = -\overset{\mathbf{I}}{W}, \quad \gamma_1 = 1 \quad \text{and} \quad \gamma_2 = 0, \tag{4.24}$$

which can be written in matrix form as

$$\mathbf{M} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -\overset{\mathbf{I}}{W} \\ 1 \\ 0 \end{pmatrix}, \tag{4.25}$$

where the matrix $\mathbf{M}$ depends only on $\alpha_1, ..., \alpha_9, I_{\boldsymbol{\tau}_1}, I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$ (the entries of $\mathbf{M}$ are given explicitly in Appendix C.1). Since β_1, β_2 and β_3 depend on $\overset{\mathbf{I}}{W}$, the above gives three linear partial differential equations for the single variable $\overset{\mathbf{I}}{W}$. This implies that if $\alpha_1, ...\alpha_9$ are unrestricted, $\overset{\mathbf{I}}{W}$ is over-prescribed. Hence, the only way to satisfy (4.25) is to restrict $\alpha_1, ..., \alpha_9$, as we show in the following section.

4.3 *The case of small initial stress*

Our goal is to expand the linearised stress (4.14) for small $\|\boldsymbol{\tau}\|$, where $\|\cdot\|$ can be the Frobenius norm, and then use linear ISRI (4.18), or equivalently (4.24), to restrict the coefficients of the stress. To achieve this we need to expand linear ISRI up to higher orders in $\|\boldsymbol{\tau}\|$, as potentially some of these terms may restrict our linearised stress. For more details on linearising functions of isotropic invariants, see (**51**).

Our approach is to take the (4.24) and expand for small $\|\boldsymbol{\tau}\|$ and neglect $\mathcal{O}(\|\boldsymbol{\tau}\|^3)$ terms. With reference to (4.23), we note that γ_1 multiplies an $\mathcal{O}(\|\boldsymbol{\tau}\|)$ term and γ_2 multiplies an $\mathcal{O}(\|\boldsymbol{\tau}\|^2)$ term. Therefore, it is only necessary to expand γ_1 up to $\mathcal{O}(\|\boldsymbol{\tau}\|)$ and γ_2 up to $\mathcal{O}(\|\boldsymbol{\tau}\|^0)$. Upon doing so, we obtain

$$\beta_1\left(\alpha_1 + 3\alpha_2 + \alpha_3 \operatorname{tr}\boldsymbol{\tau}\right) + \beta_2\left(\alpha_2 \operatorname{tr}\boldsymbol{\tau} + \alpha_3 \operatorname{tr}(\boldsymbol{\tau}^2)\right) + \beta_3\alpha_2 \operatorname{tr}(\boldsymbol{\tau}^2) = -\overset{\mathbf{I}}{W}, \tag{4.26}$$

$$\beta_1\left(3(\alpha_3 + 1) + \alpha_6 \operatorname{tr}\boldsymbol{\tau} + 2\alpha_8\right) + \beta_2\left(\alpha_1 + (\alpha_3 + 1)\operatorname{tr}\boldsymbol{\tau}\right) = 1, \tag{4.27}$$

$$2\beta_1\alpha_9 + 2\beta_2\alpha_8 + \beta_3\alpha_1 = 0. \tag{4.28}$$

Next, we expand $\alpha_1, ..., \alpha_9$ and neglect $\mathcal{O}(\|\boldsymbol{\tau}\|^2)$ terms:

$$\alpha_i = \alpha_{i0} + \alpha_{i1} \operatorname{tr}\boldsymbol{\tau} + \alpha_{i2}(\operatorname{tr}\boldsymbol{\tau})^2 + \alpha_{i3} \operatorname{tr}(\boldsymbol{\tau}^2) \quad \text{for} \quad i = 1, 2, ..., 9, \tag{4.29}$$

where the α_{ij}, for $i = 1, ...9,\ j = 0, ..., 3$, are constants. We also expand $\overset{\mathbf{I}}{W}$ up to $\mathcal{O}(\|\boldsymbol{\tau}\|^3)$:

$$\overset{\mathbf{I}}{W} = \psi_0 + \psi_1 \operatorname{tr}\boldsymbol{\tau} + \psi_2(\operatorname{tr}\boldsymbol{\tau})^2 + \psi_3 \operatorname{tr}(\boldsymbol{\tau}^2) + \psi_4(\operatorname{tr}\boldsymbol{\tau})^3 + \psi_5 \operatorname{tr}\boldsymbol{\tau} \operatorname{tr}(\boldsymbol{\tau}^2) + \psi_6 \operatorname{tr}(\boldsymbol{\tau}^3), \tag{4.30}$$

where $\psi_0, ..., \psi_6$ are constants and we immediately choose $\psi_0 = 0$ since we expect

$$\lim_{\boldsymbol{\tau}\to\mathbf{0}} \overset{\mathbf{I}}{W} = 0. \tag{4.31}$$

Upon substituting (4.30) into (4.20)–(4.22), we can obtain β_1, β_2 and β_3 expanded up to $\mathcal{O}(\|\boldsymbol{\tau}\|^2)$, $\mathcal{O}(\|\boldsymbol{\tau}\|^1)$ and $\mathcal{O}(\|\boldsymbol{\tau}\|^0)$, respectively, which can then be substituted into (4.26)–(4.28). We then solve the resulting system of equations for the parameters α_{ij} and ψ_i, where we note that the stress tensor of an initially stressed material must generalise that derived from classical linear elasticity. In other words, when $\boldsymbol{\tau} \to \mathbf{0}$ we must have

$$\delta\boldsymbol{\sigma} = \alpha_{10}\boldsymbol{\varepsilon} + \alpha_{20}\mathbf{I}\operatorname{tr}(\boldsymbol{\varepsilon}), \quad \text{where} \quad \alpha_{10} = 2\mu \quad \text{and} \quad \alpha_{20} = \lambda, \tag{4.32}$$

where λ and μ are the first and second Lamé parameters, respectively. Using (4.32), the final system of equations simplifies to the following conditions:

$$\psi_1 = 0, \qquad \psi_2 = -\frac{\lambda}{12\kappa\mu}, \qquad \psi_3 = \frac{1}{4\mu}, \tag{4.33}$$

$$\psi_4 = \frac{2\lambda^2(3\alpha_{11} - 2\alpha_{80}) + 2\lambda\mu(4\alpha_{11} + 4\alpha_{30} + 3) - 8\mu^2\alpha_{21}}{216\kappa^2\mu^2}, \tag{4.34}$$

$$\psi_5 = \frac{\lambda(2\alpha_{80} - 3\alpha_{11}) - 2\mu(\alpha_{11} + \alpha_{30} + 1)}{24\kappa\mu^2}, \qquad \psi_6 = -\frac{\alpha_{80}}{6\mu^2}, \tag{4.35}$$

$$\alpha_{80} = \frac{2\mu\alpha_{30} - 3\kappa\alpha_{11}}{2\lambda}, \tag{4.36}$$

where $\kappa = \lambda + 2\mu/3$ is the bulk modulus of the material under consideration. (4.36) relates α_{80} to λ, μ, α_{11} and α_{30}, and therefore reduces the number of free parameters in the system by one. We now use the above to write the linearised Cauchy stress in terms of the strain and initial stress:

$$\boxed{\begin{aligned}\delta\boldsymbol{\sigma} &= \boldsymbol{\tau} + \boldsymbol{\omega\tau} - \boldsymbol{\tau\omega} + \mathbf{I}\,\mathrm{tr}(\boldsymbol{\varepsilon\tau}) + 2(\mu + \mu_1\,\mathrm{tr}\,\boldsymbol{\tau})\boldsymbol{\varepsilon} + (\lambda + \lambda_1\,\mathrm{tr}\,\boldsymbol{\tau})\mathbf{I}\,\mathrm{tr}(\boldsymbol{\varepsilon}) \\ &\quad + \eta\,(\boldsymbol{\tau}\,\mathrm{tr}(\boldsymbol{\varepsilon}) + \mathbf{I}\,\mathrm{tr}(\boldsymbol{\varepsilon\tau})) + \left(\frac{\mu\eta}{\lambda} - \frac{3\kappa\mu_1}{2\lambda}\right)(\boldsymbol{\varepsilon\tau} + \boldsymbol{\tau\varepsilon}),\end{aligned}} \tag{4.37}$$

where we have renamed $\alpha_{11} = 2\mu_1$, $\alpha_{21} = \lambda_1$ and $\alpha_{30} = \eta$ and all the parameters in the equation above are constants. Equation (4.37) differs from the stress tensor first deduced in (**19**) because of the restriction given in (4.36). The parameters above may be further restricted by considerations such as strong-ellipticity (**53**, **54**), but ultimately, they can be determined by ultrasonic, indentation or hole drilling experiments.

4.3.1 *Initially stressed neo-Hookean models.* If (2.34) and (2.35) are expanded for small $\boldsymbol{\tau}$, they can be solved for K_1 and K_2, which have the same series expansion up to order one in $\boldsymbol{\tau}$:

$$K_1 = K_2 = 1 + \frac{I_{\boldsymbol{\tau}_1}}{3\kappa} + \mathcal{O}(\boldsymbol{\tau}^2). \tag{4.38}$$

Equation (4.38) can then be substituted into (4.15) and (4.16) to obtain

$$\alpha_1 = 2\mu - \frac{2(\lambda + \mu)}{3\kappa} I_{\boldsymbol{\tau}_1} + \mathcal{O}(\boldsymbol{\tau}^2), \qquad \alpha_2 = \lambda - \frac{\lambda}{3\kappa} I_{\boldsymbol{\tau}_1} + \mathcal{O}(\boldsymbol{\tau}^2), \tag{4.39}$$

for the first model, and

$$\alpha_1 = 2\mu - \frac{2(\lambda + \mu)}{3\kappa} I_{\boldsymbol{\tau}_1} + \mathcal{O}(\boldsymbol{\tau}^2), \qquad \alpha_2 = \lambda + \frac{2\lambda}{3\kappa} I_{\boldsymbol{\tau}_1} + \mathcal{O}(\boldsymbol{\tau}^2). \tag{4.40}$$

for the second. Therefore, for both models, we have

$$\alpha_{10} = 2\mu, \quad \alpha_{11} = -\frac{2(\lambda + \mu)}{3\kappa}, \quad \alpha_{20} = \lambda \quad \alpha_{30} = -1, \quad \text{and} \quad \alpha_{80} = 1, \tag{4.41}$$

which satisfy (4.36), as required. The linearised stress tensors associated with the two models are

$$\delta\boldsymbol{\sigma}_{\text{GSC1}} = \boldsymbol{\tau} + \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\tau}\,\text{tr}(\boldsymbol{\varepsilon}) + 2\left(\mu - \frac{\lambda+\mu}{3\kappa}\,\text{tr}\,\boldsymbol{\tau}\right)\boldsymbol{\varepsilon} \tag{4.42}$$

$$+\left(\lambda - \frac{\lambda}{3\kappa}\,\text{tr}\,\boldsymbol{\tau}\right)\mathbf{I}\,\text{tr}(\boldsymbol{\varepsilon}) + \boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}, \tag{4.43}$$

and

$$\delta\boldsymbol{\sigma}_{\text{GSC2}} = \boldsymbol{\tau} + \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\tau}\,\text{tr}(\boldsymbol{\varepsilon}) + 2\left(\mu - \frac{\lambda+\mu}{3\kappa}\,\text{tr}\,\boldsymbol{\tau}\right)\boldsymbol{\varepsilon} \tag{4.44}$$

$$+\left(\lambda + \frac{2\lambda}{3\kappa}\,\text{tr}\,\boldsymbol{\tau}\right)\mathbf{I}\,\text{tr}(\boldsymbol{\varepsilon}) + \boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}. \tag{4.45}$$

5. Discussion

Many constitutive choices in the literature of the form $W := W(\mathbf{F}, \boldsymbol{\tau})$ do not satisfy the ISRI restrictions (2.12) and (4.18) presented in this article. In Section 2.1, we gave an example of how these constitutive choices may lead to unphysical behaviour even for simple deformations such as uniaxial extension. This is also true of more complex deformations. Taking an example from biomechanics, where residual stresses play a crucial role, suppose we wish to model the mechanics of an arterial wall that supports an internal pressure. Let us choose two different reference configurations: first, the *unloaded configuration* where the fluid in the artery has been removed, and second, the *opening angle configuration* (**60**, **30**) where the fluid has been removed and the artery has been cut along its axis. Both these configurations are subject to no external loads, but there will be less (and differently distributed) internal stress in the opening angle configuration. If we use a strain energy function $W(\mathbf{F}, \boldsymbol{\tau})$ that does not satisfy ISRI, then each of the two reference configurations will lead to a different stress distribution in the intact, inflated configuration of the arterial wall. We therefore cannot believe the preditions from either reference configuration since a physically correct model should *not* give different results due to an arbitrary choice of reference configuration.

By using ISRI, we were able to derive a restricted form for the linear elastic stress tensor (4.37) in the case of small initial stress. This reduced form may ultimately improve material characterisation based on ultrasonic and indentation experiments. Many studies (see (**19**) and the references therein) have confirmed that a linearised stress tensor of the form given in (4.37) is well-suited to fitting experimental data.

One outstanding problem for metals (**55**), biological soft tissues and other materials (**56**) is the difficulty in differentiating between the effects of structural anisotropy (**57**) and anisotropy caused by initial stress. The linear form of ISRI given in (4.18) will help to differentiate between these effects, as it dictates a specific dependency of the elastic stress on the initial stress. Nevertheless, future work should focus on developing the consequences of ISRI for materials with structural anisotropy. This will be particularly important for collagenous soft tissues, which are known to be structurally anisotropic due to the presence of collagen fibres (**58**, **59**). Initial stresses in soft tissues can be significant (**60**, **25**, **46**), so assuming a small initial stress may not give accurate predictions. Currently, the internal stress in soft tissues is often measured by excising a sample and then estimating its initial deformation from a theoretically stress-free configuration. To measure stress *in-vivo*, non-invasive techniques need to be improved. Ultrasound techniques are among the most suitable and promising methods for measuring initial stress (**61**, **62**), and the ISRI restrictions could ultimately improve them.

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Data access:

Data access statement: All the data associated with this work are contained within the article.

Author contributions:

T.S. and A.L.G. drafted the article and carried out and verified all the calculations. A.L.G. conceived of the study. P.C. commented on and edited the article. All authors gave final approval for publication.

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APPENDIX A

A.1 Deduction of the strain energy function W_{GCD}

The strain energy function (2.22) was first derived in (**31**). Here, an alternative derivation is presented by considering deformations of an incompressible neo-Hookean material from a stress-free configuration $\mathcal{B}_0$ to the stressed configurations $\mathcal{B}$ and $\overline{\mathcal{B}}$ (see Fig. A1 and compare with Fig. 2).

The neo-Hookean strain energy function is given by

$$W_{\mathrm{NH}} = \mu(I_1 - 3), \tag{A.1}$$

where μ is the ground state shear modulus of the material under consideration. Upon substituting (A.1) into (2.18) with $W_{I_3} = 0$ (because the material is incompressibile) and then taking $\mathbf{F} = \mathbf{F}_0$ and $\mathbf{F} = \mathbf{F}_1$, it follows that

$$\boldsymbol{\tau} = \mu\mathbf{B}_0 - p_0\mathbf{I} \quad \text{and} \quad \overline{\boldsymbol{\sigma}} = \mu\mathbf{B}_1 - p_1\mathbf{I}, \tag{A.2}$$

where $\mathbf{B}_0 = \mathbf{F}_0\mathbf{F}_0^{\mathrm{T}}$, $\mathbf{B}_1 = \mathbf{F}_1\mathbf{F}_1^{\mathrm{T}}$ and p_0 and p_1 are the Lagrange multipliers associated with the two respective deformations. By rearranging $(\mathrm{A}.2)_1$ and taking the determinant of both sides, the following is obtained:

$$\det(\mu\mathbf{B}_0) = \det(\boldsymbol{\tau} + p_0\mathbf{I}) \quad \Leftrightarrow \quad \mu^3 = p_0^3 + p_0^2 I_{\boldsymbol{\tau}1} + p_0 I_{\boldsymbol{\tau}2} + I_{\boldsymbol{\tau}3}, \tag{A.3}$$

where $\det(\mathbf{B}_0) = 1$ because the material is incompressible. Only one of the three roots of the above polynomial is physically meaningful (**31**) and it is given by (2.23). Using $\mathbf{F}_1 = \overline{\mathbf{F}}\mathbf{F}_0$, $(\mathrm{A}.2)_2$ gives

$$\overline{\boldsymbol{\sigma}} = \mu\overline{\mathbf{F}}\mathbf{B}_0\overline{\mathbf{F}}^{\mathrm{T}} - p_1\mathbf{I}. \tag{A.4}$$

The aim is to derive an initially stressed strain energy function that gives (A.4) with $\mathcal{B}$ as the reference configuration. For simplicity, it is assumed that the strain energy function depends only upon I_1, J_1 and the

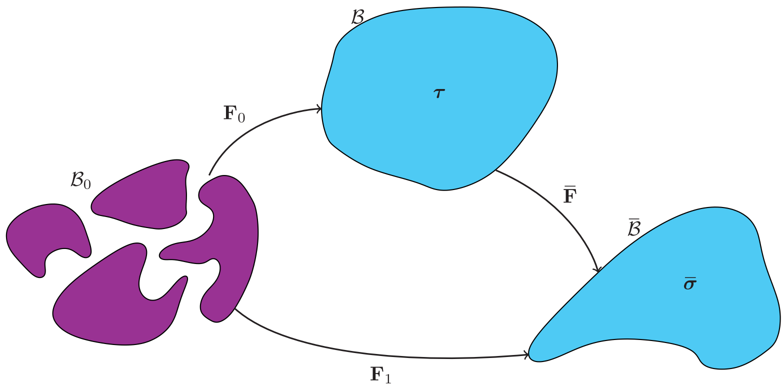

Fig. A1 Deformation of an incompressible neo-Hookean material from a stress-free configuration $\mathcal{B}_0$ to the stressed configurations $\mathcal{B}$ and $\overline{\mathcal{B}}$

three initial stress invariants I_{τ_1}, I_{τ_2} and I_{τ_3}. Making this assumption and substituting $\mathbf{F} = \overline{\mathbf{F}}$ into (2.18) with $W_{I_3} = 0$, it follows that

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\overline{\mathbf{F}}, \boldsymbol{\tau}) = 2W_1\overline{\mathbf{B}} + 2W_{J_1}\overline{\mathbf{F}}\boldsymbol{\tau}\overline{\mathbf{F}}^{\mathrm{T}} - \overline{p}\mathbf{I} \tag{A.5}$$

$$= 2W_1\overline{\mathbf{B}} + 2W_{J_1}(\mu\overline{\mathbf{F}}\mathbf{B}_0\overline{\mathbf{F}}^{\mathrm{T}} - p_0\overline{\mathbf{B}}) - \overline{p}\mathbf{I}. \tag{A.6}$$

For equation (A.6) to be equivalent to (A.4), the following equations must be satisfied:

$$2W_1 = p_0, \qquad 2W_{J_1} = 1, \qquad \overline{p} = p_1. \tag{A.7}$$

The third of these equations does not tell us anything about the required functional form of W; however, upon solving the first two, the following is obtained:

$$W = \frac{1}{2}(p_0(I_{\tau_1}, I_{\tau_2}, I_{\tau_3})I_1 + J_1) + f(I_{\tau_1}, I_{\tau_2}, I_{\tau_3}), \tag{A.8}$$

where f is an arbitrary function of I_{τ_1}, I_{τ_2} and I_{τ_3}. Upon choosing $f(I_{\tau_1}, I_{\tau_2}, I_{\tau_3}) = -\frac{3}{2}\mu$, the final form of the strain energy function (2.22) is obtained. This choice ensures that the energy derived using the initially stressed strain energy function is the same as that obtained by considering a direct deformation of a neo-Hookean material from the stress-free configuration.

All that remains is to prove that, when using W_{GCD}, the third equation of (A.7) holds. Equations $(\mathrm{A}.2)_1$ and (A.4) can be rearranged to give

$$p_0\mathbf{I} = \mu\mathbf{B}_0 - \boldsymbol{\tau} \quad \text{and} \quad p_1\mathbf{I} = \mu\overline{\mathbf{F}}\mathbf{B}_0\overline{\mathbf{F}}^{\mathrm{T}} - \overline{\boldsymbol{\sigma}}, \tag{A.9}$$

respectively. Multiplying the first of these equations on the left by $\overline{\mathbf{F}}$ and on the right by $\overline{\mathbf{F}}^{\mathrm{T}}$, and upon substituting equation (A.8) into equation (A.5) and equation (A.5) into equation $(\mathrm{A}.9)_2$, we obtain

$$p_0\overline{\mathbf{B}} = \mu\overline{\mathbf{F}}\mathbf{B}_0\overline{\mathbf{F}}^{\mathrm{T}} - \overline{\mathbf{F}}\boldsymbol{\tau}\overline{\mathbf{F}}^{\mathrm{T}} \tag{A.10}$$

and

$$p_1\mathbf{I} = \mu\overline{\mathbf{F}}\mathbf{B}_0\overline{\mathbf{F}}^{\mathrm{T}} - p_0\overline{\mathbf{B}} + \overline{p}\mathbf{I} - \overline{\mathbf{F}}\boldsymbol{\tau}\overline{\mathbf{F}}^{\mathrm{T}}, \tag{A.11}$$

respectively. Then substituting (A.10) into (A.11), we obtain

$$p_1\mathbf{I} = \overline{p}\mathbf{I} \quad \Rightarrow \quad p_1 = \overline{W}p, \tag{A.12}$$

as required.

APPENDIX B

B.1 Tensor identities

The Cayley–Hamilton theorem allows us to determine which tensors are independent. It states that any 3×3 tensor $\mathbf{A}$ satisfies

$$\mathbf{A}^3 - I_{\mathbf{A}_1}\mathbf{A}^2 + I_{\mathbf{A}_2}\mathbf{A} - I_{\mathbf{A}_3}\mathbf{I} = \mathbf{0}, \tag{B.1}$$

where $I_{\mathbf{A}_1}$, $I_{\mathbf{A}_2}$ and $I_{\mathbf{A}_3}$ are the invariants of $\mathbf{A}$ analagous to $I_{\boldsymbol{\tau}_1}$, $I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$ for $\boldsymbol{\tau}$. From (B.1), we can see that any power of $\boldsymbol{\tau}$ higher than two can be rewritten in terms of $\boldsymbol{\tau}^2$, $\boldsymbol{\tau}$, $\mathbf{I}$ and the invariants $I_{\boldsymbol{\tau}_1}$, $I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$.

We will now show that $\operatorname{tr}(\boldsymbol{\tau}^2\boldsymbol{\varepsilon})$ and $\boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau}$, $\boldsymbol{\tau}^2\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau}^2$ can be written as combinations of terms already present in (4.8). First substitute $\mathbf{A} = \boldsymbol{\varepsilon} + \gamma\boldsymbol{\tau}$ in (B.1), where γ is an arbitrary scalar. Since the resulting equation must hold for every γ, each coefficient multiplying a different power of γ must be zero individually. The term multiplying γ^2 is given by

$$\begin{aligned}\boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\varepsilon}\boldsymbol{\tau}^2 + \boldsymbol{\tau}^2\boldsymbol{\varepsilon} - (\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon})I_{\boldsymbol{\tau}_1} - \boldsymbol{\tau}^2\operatorname{tr}\boldsymbol{\varepsilon} + \boldsymbol{\tau}(I_{\boldsymbol{\tau}_1}\operatorname{tr}\boldsymbol{\varepsilon} - \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau})) + \boldsymbol{\varepsilon}I_{\boldsymbol{\tau}_2}&\\ + \mathbf{I}(I_{\boldsymbol{\tau}_1}\operatorname{tr}(\boldsymbol{\tau}\boldsymbol{\varepsilon}) - I_{\boldsymbol{\tau}_2}\operatorname{tr}\boldsymbol{\varepsilon} - \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{\tau}^2)) = 0.&\end{aligned} \tag{B.2}$$

By taking the trace of both sides of this equation (and using the properties $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{B}$ and $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$) we establish that $\operatorname{tr}(\boldsymbol{\tau}^2\boldsymbol{\varepsilon})$ is indeed a combination of the terms already present in (4.8). The same can then be said for $\boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau}$ directly from (B.2), and for $\boldsymbol{\tau}^2\boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}\boldsymbol{\tau}^2$ by multiplying (B.2) on the left by $\boldsymbol{\tau}$.

APPENDIX C

C.1 The entries of the matrix M

The entries of the matrix $\mathbf{M}$ are as follows:

$$\mathrm{M}_{11} = \alpha_1 + 3\alpha_2 + \alpha_3\operatorname{tr}\boldsymbol{\tau}, \qquad \mathrm{M}_{12} = \alpha_2\operatorname{tr}\boldsymbol{\tau} + \alpha_3\operatorname{tr}(\boldsymbol{\tau}^2) + 2\alpha_9 I_{\boldsymbol{\tau}_3}, \tag{C.1}$$

$$\mathrm{M}_{13} = \alpha_2\operatorname{tr}(\boldsymbol{\tau}^2) + \alpha_3\operatorname{tr}(\boldsymbol{\tau}^3) + 2\alpha_8 I_{\boldsymbol{\tau}_3} + 2\alpha_9 I_{\boldsymbol{\tau}_1}I_{\boldsymbol{\tau}_3}, \tag{C.2}$$

$$\mathrm{M}_{21} = 3(\alpha_3 + 1) + \alpha_6\operatorname{tr}\boldsymbol{\tau} + 2\alpha_8, \tag{C.3}$$

$$\mathrm{M}_{22} = \alpha_1 + (\alpha_3 + 1)\operatorname{tr}\boldsymbol{\tau} + \alpha_6\operatorname{tr}(\boldsymbol{\tau}^2) - 2\alpha_9 I_{\boldsymbol{\tau}_2}, \tag{C.4}$$

$$M_{23} = (\alpha_3 + 1)\,\mathrm{tr}(\boldsymbol{\tau}^2) + \alpha_6\,\mathrm{tr}(\boldsymbol{\tau}^3) - 2\alpha_8 I_{\boldsymbol{\tau}_2} + 2\alpha_9 (I_{\boldsymbol{\tau}_3} - I_{\boldsymbol{\tau}_1} I_{\boldsymbol{\tau}_2}), \tag{C.5}$$

$$M_{31} = 2\alpha_9, \qquad M_{32} = 2\alpha_8 + 2\alpha_9\,\mathrm{tr}\,\boldsymbol{\tau}, \tag{C.6}$$

$$M_{33} = \alpha_1 + 2\alpha_8\,\mathrm{tr}\,\boldsymbol{\tau} + 2\alpha_9\,\mathrm{tr}(\boldsymbol{\tau}^2). \tag{C.7}$$